Maximization and Strictly Increasing Transformation

The utility function of a consumer is given by $U(x_1, x_2) = x_1^{1/3} x_2^{1/2}$ where x_1 and x_2 represent the quantities of goods 1 and 2 consumed in a given period of time. Let p_1 and p_2 be the unit prices of each of the goods and M the amount of money that the individual will spend on the acquisition of both goods.

1. Calculate the quantity to consume of each of the goods as a function of the parameters p_1, p_2 and M if the utility is to be maximized.

2. Obtain the indirect objective function and calculate the partial derivatives. Show that an increase in M produces an increase in maximum utility and an increase in p_1 or p_2 a decrease in maximum utility.

3. Solve the previous point taking as objective function $g(x_1, x_2) = ln(U(x_1, x_2))$ What do you observe?

Solution

1. The constraint we have is $x_1p_1 + x_2p_2 = M$. We construct the Lagrangian:

$$L = x_1^{1/3}x_2^{1/2} + \lambda(M - x_1p_1 - x_2p_2)$$

$$L'_{x_1} = \frac{1}{3}x_1^{-2/3}x_2^{1/2} - \lambda p_1 = 0$$

$$L'_{x_2} = \frac{1}{2}x_1^{1/3}x_2^{-1/2} - \lambda p_2 = 0$$

$$L'_{\lambda} = M - x_1p_1 - x_2p_2 = 0$$

From the first two equations, we clear λ

$$\frac{\frac{1}{3}x_1^{-2/3}x_2^{1/2}}{p_1} = \lambda$$

$$\frac{\frac{1}{2}x_1^{1/3}x_2^{-1/2}}{p_2} = \lambda$$

We equate the two equations:

$$\frac{\frac{1}{3}x_1^{-2/3}x_2^{1/2}}{p_1} = \frac{\frac{1}{2}x_1^{1/3}x_2^{-1/2}}{p_2}$$

Reordering:

$$\frac{\frac{1}{3}x_1^{-2/3}x_2^{1/2}}{\frac{1}{2}x_1^{1/3}x_2^{-1/2}} = \frac{p_1}{p_2}$$

This condition indicates that the slopes of the level curves must be equal, that is, the MRS with the price ratio. Now we clear one variable in terms of the other:

$$\frac{x_2}{x_1} = \frac{p_1}{p_2}\frac{3}{2}$$

$$x_2 = x_1\frac{p_1}{p_2}\frac{3}{2}$$

We insert this into the third equation of the first-order conditions:

$$M - x_1p_1 - p_2\left[x_1\frac{p_1}{p_2}\frac{3}{2}\right] = 0$$

Now I clear x_1

$$M = x_1p_1 + \left[x_1\frac{p_1}{1}\frac{3}{2}\right]$$

$$M = x_1p_1\left[1 + \frac{3}{2}\right]$$

$$\frac{M}{p_1} = x_1\frac{5}{2}$$

$$\frac{2}{5}\frac{M}{p_1} = x_1^*$$

We insert this into the x_2 equation to obtain the other optimal quantity:

$$x_2 = \frac{2}{5}\frac{M}{p_1}\frac{p_1}{p_2}\frac{3}{2}$$

$$x_2 = \frac{M}{p_2}\frac{3}{5}$$

Finally, we obtain the value of the Lagrange multiplier:

$$\lambda = \frac{\frac{1}{3}\left(\frac{2}{5}\frac{M}{p_1}\right)^{-2/3}\left(\frac{M}{p_2}\frac{3}{5}\right)^{1/2}}{p_1}$$

$$\lambda^* = (0.4756)M^{-1/6}p_1^{-1/3}p_2^{-1/2}$$

2. Now we obtain the indirect utility function:

$$V(p_1, p_2, M) = U(x_1^* x_2^*) = \left(\frac{M}{p_1}\frac{2}{5}\right)^{(1/3)}\left(\frac{M}{p_2}\frac{3}{5}\right)^{(1/2)}$$

Then:

$$V(p_1, p_2, M) = M^{5/2}p_1^{-1/3}p_2^{-1/2}\left(\frac{2}{5}\right)^{(1/3)}\left(\frac{3}{5}\right)^{(1/2)}$$

We calculate the derivatives:

$$V'_M = \frac{5}{2}M^{3/2}p_1^{-1/3}p_2^{-1/2}\left(\frac{2}{5}\right)^{(1/3)}\left(\frac{3}{5}\right)^{(1/2)} > 0$$

$$V'_{p_1} = -\frac{1}{3}M^{5/2}p_1^{-4/3}p_2^{-1/2}\left(\frac{2}{5}\right)^{(1/3)}\left(\frac{3}{5}\right)^{(1/2)} < 0$$

$$V'_{p_2} = -\frac{1}{2}M^{5/2}p_1^{-1/3}p_2^{-3/2}\left(\frac{2}{5}\right)^{(1/3)}\left(\frac{3}{5}\right)^{(1/2)} < 0$$

3. We assemble the Lagrangian again:

$$L = \frac{1}{3}ln(x_1) + \frac{1}{2}ln(x_2) + \lambda(M - x_1p_1 - x_2p_2)$$

The first order conditions:

$$L'_{x_1} = \frac{1}{3x_1} - \lambda p_1 = 0$$

$$L'_{x_2} = \frac{1}{2x_2} - \lambda p_2 = 0$$

$$L'_{\lambda} = M - x_1p_1 - x_2p_2 = 0$$

From the first two equations:

$$\frac{1}{3x_1p_1} = \lambda$$

$$\frac{1}{2x_2p_2} = \lambda$$

We equalize:

$$\frac{1}{2x_2p_2} = \frac{1}{3x_1p_1}$$

We solve for:

$$x_1\frac{3}{2}\frac{p_1}{p_2} = x_2$$

We insert this into the third equation of the first order conditions:

$$M - x_1p_1 - p_2\left[x_1\frac{p_1}{p_2}\frac{3}{2}\right] = 0$$

Now I solve for x_1

$$M = x_1 p_1 + \left[x_1 \frac{p_1}{1}\frac{3}{2}\right]$$

$$M = x_1 p_1 \left[1 + \frac{3}{2}\right]$$

$$\frac{M}{p_1} = x_1 \frac{5}{2}$$

$$\frac{2}{5}\frac{M}{p_1} = x_1^*$$

We insert this into the x_2 equation to get the other optimal quantity:

$$x_2 = \frac{2}{5}\frac{M}{p_1}\frac{p_1}{p_2}\frac{3}{2}$$

$$x_2 = \frac{M}{p_2}\frac{3}{5}$$

Finally, we find the value of the Lagrange multiplier:

$$\lambda = \frac{1}{3\left(\frac{2}{5}\frac{M}{p_1}\right)p_1}$$

$$\lambda^* = (0.4)M^{-1}p_1^{-2}$$

The optimal values are the same as before, this is due to the fact that we applied a strictly increasing transformation (natural logarithm). However, the value of λ now changes:

$$\lambda = \frac{1}{3p_1\left[\frac{2}{5}\frac{M}{p_1}\right]}$$

$$\lambda = \frac{6}{5}M^{-1}p_1^{-1}$$

We calculate the second order conditions:

$$L''_{x_1x_1} = -\frac{1}{3x_1^2}$$

$$L''_{x_2x_2} = -\frac{1}{2x_2^2}$$

$$L''_{x_1x_2} = L''_{x_2x_1} = 0$$

$$g'x_1 = p_1$$

$$g'x_2 = p_2$$

We assemble the bordered Hessian:

$$\bar{H} = \begin{pmatrix} 0 & g'x & g'y \\ g'x & L''_{xx} & L''_{xy} \\ g'y & L''_{yx} & L''_{yy} \end{pmatrix} = \begin{pmatrix} 0 & p_1 & p_2 \\ p_1 & -\frac{1}{3x_1^2} & 0 \\ p_2 & 0 & -\frac{1}{2x_2^2} \end{pmatrix}$$

We calculate the determinant:

$$-p_1 \begin{vmatrix} p_1 & p_2 \\ 0 & -\frac{1}{2x_2^2} \end{vmatrix} + p_2 \begin{vmatrix} p_1 & p_2 \\ -\frac{1}{3x_1^2} & 0 \end{vmatrix} = p_1(\frac{p_1}{2x_2^2}) + p_2(\frac{p_2}{3x_1^2}) > 0$$

Since the determinant of the bordered Hessian is positive, we are in the presence of a maximum.